386. Proposed by HERBERT N. CARLETON, Newberry, Mass.

C is a fixed point on the perpendicular bisector of the line segment AB. Locate a point D also on this bisector, such that $AD + BD + DC$ shall be a minimum.

MECHANICS.

309. Proposed by JOS. B. REYNOLDS, Lehigh University.

The tangent at one cusp of a vertical, three arched hypocycloid is horizontal, and a particle will just slide under gravity from the upper cusp to this cusp. Find the equation which the coefficient of friction must satisfy.

310. Proposed by EMMA M. GIBSON, Drury College.

A particle movable on a smooth spherical surface of radius a is projected along a horizontal great circle with a velocity v which is great compared with $\sqrt{2ga}$. Prove that its path lies between this great circle and a parallel circle whose plane is approximately at a depth $2ga^2/v^2$ below the center.

From Lamb's *Dynamics*, p. 334, Ex. 3.

SOLUTIONS OF PROBLEMS.

Note.—When several persons send in solutions for the same problem, the committee naturally and, we think, properly select for publication that one which is not only correct mathematically but is written out in the best form for publication. They must either do this or else, if they select solutions which are in poor form, in order to give as many solvers as possible a chance, they must write these solutions over to save them from rejection by the Managing Editor as bad copy. The task of *putting solutions into acceptable shape for the printer* is one which the members of the committee do not relish,—and who can blame them? This will explain why some names appear more frequently at the head of the solutions than do others, even though several may have solved the same problem. See the suggestions for preparing solutions published in several previous issues. MANAGING EDITOR.

ALGEBRA.

409. Proposed by C. E. GITHENS, Wheeling, W. Va.

Find integral values for the edges of a rectangular parallelopiped so that its diagnoal shall be rational.

REMARKS BY W. C. EELLS, U. S. Naval Academy.

In the February, 1915, issue of the MONTHLY (pp. 60–61) Artemas Martin criticizes my solution of this problem in the October, 1914, issue, stating that I have solved a different problem from the one proposed, and that I claimed that a certain rational parallelopiped was the smallest possible one, whereas he exhibits four others that are smaller.

Since the problem reads "Find integral values" and not "Find *all* integral values, etc." it was in order to impose the condition $x^2 + y^2 = k^2$ or any other condition so long as integral solutions of the equation $x^2 + y^2 + z^2 = a^2$ were found. I showed two general methods of solution, each giving an infinite number of *prime* integral solutions, but did not state nor even suppose that I had found all possible solutions. I fail to see, however, how I solved a *different* problem from the one proposed.

Under my first method, as an example, I gave the solution $(x, y, z, a) = (4, 3, 12, 13)$ as the smallest rational parallelopiped, and it should have been sufficiently

evident that what I meant was that this was the smallest one found by this method,—not the smallest one possible. The latter would have been a rash and unwarranted statement, when I was not professing to give *all* solutions. Mr. Martin cites the four solutions (1, 2, 2, 3), (2, 3, 6, 7), (1, 4, 8, 9), (2, 6, 9, 11) to prove that (3, 4, 12, 13) is not the smallest. He might have cited two more, (4, 4, 7, 9) and (6, 6, 7, 11).

Mr. Martin gives the equation

$$a^2 + b^2 + \left(\frac{a^2 + b^2 - c^2}{2c}\right)^2 = \left(\frac{a^2 + b^2 + c^2}{2c}\right)^2,$$

"true for all values of a, b, c," from which to derive solutions of the equation $x^2 + y^2 + z^2 = d^2$. This identity is true but quite useless unless further and quite elaborately qualified. For many values of a, b, c, it gives z and d fractional values in violation of the conditions of the problem. Mr. Martin gives no restrictions on values of a, b, c, the obvious implication being that one is free to assign values to them at pleasure. For $a = 5$, $b = 3$, $c = 1$, it gives $z = 33/2$, $d = 35/2$. As a matter of fact there are *no* integral solutions possible for a and b both odd. For let $a = 2m + 1$, $b = 2n + 1$, and let c be even, $c = 2p$. Then

$$z \equiv \frac{4m^2 + 4n^2 + 4m + 4n + 2 - 4p}{4p} \equiv \frac{4K + 1}{2p},$$

evidently impossible as an integral solution. Again let c be odd, $c = 2p + 1$. As before

$$z = \frac{4K + 1}{2(2p + 1)},$$

also impossible. Thus there are no solutions for a and b both odd. Similarly it can easily be shown that there are *no* solutions for c even, when a and b are even-odd or odd-even, but only when a and b are even-even, and not always then. When c is odd there are no solutions if a and b are both even, but only when a and b are even-odd or odd-even. It requires further careful discrimination to properly restrict the form of a and b in the two cases indicated as sometimes yielding solutions.

Soon after submitting the solution to which Mr. Martin objects I set to work to devise a method for finding *all* solutions, worked one out along the lines suggested above, and calculated all possible rational prime solutions, 74 in number for which the diagonal is less than 50. (My previous methods had given only three such.) But on submitting this to the MONTHLY some two months ago Professor R. D. Carmichael kindly called my attention to an elegant solution of the same problem by V. A. Le Besque which was published in the *Comptes Rendus* in 1868 (Vol. 66, pp. 396–398). Le Besque's method was so superior to the one I had devised (as well as to the one proposed by Mr. Martin) that I did not think mine worth publishing and did nothing further with it.

Since the problem has come up again, however, it may be of interest and

value to publish in the MONTHLY Le Besque's method. He gives the following identity,

$$(\alpha^2+\beta^2+\gamma^2+\delta^2)^2 \equiv (\alpha^2+\beta^2-\gamma^2-\delta^2)^2+4(\alpha\gamma+\beta\delta)^2+4(\alpha\delta-\beta\gamma)^2.$$

Since every integral number n can be expressed in the form

$$n=\alpha^2+\beta^2+\gamma^2+\delta^2, \quad (\alpha, \beta, \gamma, \delta = 0, 1, 2, 3, \cdots)$$

this affords an easy and rapid means of finding integral sides for any integral diagonal n. Le Besque's identity needs certain restrictions which need not be stated here, on the form and relative size of α, β, γ, δ, to avoid duplication and results not relatively prime. If we put $\delta = 0$ it affords a still more rapid method of finding an indefinite number of solutions, although of course not all of them. For $\beta = \delta = 0$ it gives the well-known right triangle solutions, $\alpha^2+\gamma^2$, $\alpha^2-\gamma^2$, $4\alpha\gamma$.

FURTHER REMARKS BY HERBERT N. CARLETON, Newberry, Mass.

Mr. Martin's formula can be much simplified and brought to a form in which two numbers representing two of its sides can be directly derived.

Thus, let a, b, c be the three edges. Then since $(a+b)^2 \equiv a^2+2ab+b^2$, it is only necessary to choose a and b so that $2ab = \square$. When such values of a and b are determined, the third edge, $c = \sqrt{2ab}$, and we have

$$a^2+b^2+c^2=\square.$$

Since $2ab = a^2\cdot\dfrac{2b}{a}$, if $a^2 = \dfrac{2b}{a}$, or $b = \frac{1}{2}a^3$, an integer, the conditions are fulfilled. From this it is seen that a^3 must be even and therefore a must be even.

Hence, by taking a equal to any even number, $b = \frac{1}{2}a^3$, and $c = \sqrt{2ab}$, we get numbers satisfying the conditions of the problem.

Note.—Each of these methods of solution has value and each satisfies the conditions of the problem, and none of them, it appears, will by any direct method include all possible solutions. Such a solution, so far as we know, does not exist. EDITORS.

420. Proposed by ELBERT H. CLARKE, Purdue University.

Given the infinite series,

$$\frac{a}{r}+\frac{b}{r^2}+\frac{a+b}{r^3}+\frac{a+2b}{r^4}+\frac{2a+3b}{r^5}+\cdots,$$

in which a and b are any numbers and where each numerator after the second is the sum of the two preceding numerators. To find the region of convergence and the sum of the series.

This problem is a generalization of one solved in the January (1914) number of the MONTHLY.

SOLUTION BY MRS. ELIZABETH B. DAVIS, U. S. Naval Observatory.

For $r \leqq 1$, the series is divergent. For $r > 1$, the series is convergent. Let

$$S=\frac{a}{r}+\frac{b}{r^2}+\frac{a+b}{r^3}+\frac{a+2b}{r^4}+\cdots.$$